Proper-Complex Gaussian Random Vectors

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Review. The received signal contains a noise $N(t)$ often modeled by an **additive white Gaussian noise (AWGN)**. To understand it, we started to review a pre-requisite of this course, *Probability, Random Variables, and Random Processes*.

- statistical independence of multiple events, and multiple random variables
- correlation, covariance, and correlation coefficient of two real-valued random variables, and two complex-valued random variables.
- real-valued Gaussian random vectors
  * characteristic function of a real-valued random variable
  * real-valued Gaussian random vectors

Preview. We will continue examining the notion of **Gaussianity**.

- complex-valued Gaussian random vectors
- proper-complex Gaussian random vectors
- Complex-Valued Gaussian Random Vector

- Definition. Suppose that $\mathbf{X} \in \mathbb{R}^N$ and $\mathbf{Y} \in \mathbb{R}^N$ are real-valued random vectors such that $F_{\mathbf{X},\mathbf{Y}}(x, y)$ exists. Then,

$$\mathbf{Z} \triangleq \mathbf{X} + j\mathbf{Y}$$

is called a complex-valued random vector of length $N$.

* abuse of notation
* A complex-valued random vector $\mathbf{Z} \in \mathbb{C}^N$ is completely characterized by the joint CDF/PDF/CF of Re{\mathbf{Z}} and Im{\mathbf{Z}}.
* A complex-valued random vector of length $N$ is equivalent to a real-valued random vector of length $2N$. 
- Definition.

A random vector \( Z \triangleq X + jY \in \mathbb{C}^N \) is a **complex-valued Gaussian random vector** if the real-valued random vectors \( X \in \mathbb{R}^N \) and \( Y \in \mathbb{R}^N \) are jointly Gaussian, i.e., there exists a length-\( 2N \) real-valued vector \( \mu \) and a size-\( 2N \) symmetric positive semi-definite real-valued matrix \( C \) such that

\[
\begin{bmatrix} X \\ Y \end{bmatrix} \sim \mathcal{N}(\mu, C).
\]

*Q. Do we always have to convert a complex Gaussian random vector to a real Gaussian random vector?*
Theorem. A complex Gaussian random vector \( \mathbf{Z} \in \mathbb{C}^N \) is completely characterized by

* the mean vector \( \mu_{\mathbf{Z}} \triangleq \mathbb{E}[\mathbf{Z}] \),

* the covariance matrix \( C_{\mathbf{Z}\mathbf{Z}} \triangleq \mathbb{E}
\left[(\mathbf{Z} - \mu_{\mathbf{Z}})(\mathbf{Z} - \mu_{\mathbf{Z}})^H\right] \), and

* the pseudo-covariance matrix \( \tilde{C}_{\mathbf{Z}\mathbf{Z}} \triangleq \mathbb{E}
\left[(\mathbf{Z} - \mu_{\mathbf{Z}})(\mathbf{Z} - \mu_{\mathbf{Z}})^T\right] \).

Proof.

* We know that \( \mathbf{Z} \) is completely characterized by the characteristic function \( \Phi_{\mathbf{W}}(\omega) \) of

\[
\mathbf{W} \triangleq \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix} \triangleq \begin{bmatrix} \text{Re}\{\mathbf{Z}\} \\ \text{Im}\{\mathbf{Z}\} \end{bmatrix} \in \mathbb{R}^{2N}
\]

and that \( \Phi_{\mathbf{W}}(\omega) \) is determined by \( \mu_{\mathbf{W}} \) and \( C_{\mathbf{WW}} \). Thus, we are to show that \( \mu_{\mathbf{W}} \) and \( C_{\mathbf{WW}} \) are determined by \( \mu_{\mathbf{Z}} \), \( C_{\mathbf{ZZ}} \), and \( \tilde{C}_{\mathbf{ZZ}} \).
* First, it can be easily seen that

\[ \mu_W = \begin{bmatrix} \mathbb{E}[X] \\ \mathbb{E}[Y] \end{bmatrix} = \begin{bmatrix} \text{Re}\{\mu_Z\} \\ \text{Im}\{\mu_Z\} \end{bmatrix} \]

is determined by \( \mu_Z \).

* Second, note that

\[ C_{WW} = \begin{bmatrix} C_{XX} & C_{XY} \\ C_{YX} & C_{YY} \end{bmatrix}. \]

Since

\[ C_{ZZ} = \mathbb{E}\left[ \left( (X - \mu_X) + j(Y - \mu_Y) \right) \left( (X - \mu_X)^T - j(Y - \mu_Y)^T \right) \right] = (C_{XX} + C_{YY}) + j(-C_{XY} + C_{YX}) \]

and

\[ \tilde{C}_{ZZ} = \mathbb{E}\left[ \left( (X - \mu_X) + j(Y - \mu_Y) \right) \left( (X - \mu_X)^T + j(Y - \mu_Y)^T \right) \right] = (C_{XX} - C_{YY}) + j(C_{XY} + C_{YX}), \]
we have

\[ C_{XX} = \frac{1}{2} \left[ \text{Re}\{C_{ZZ}\} + \text{Re}\{\tilde{C}_{ZZ}\} \right] \]

\[ C_{YY} = \frac{1}{2} \left[ \text{Re}\{C_{ZZ}\} - \text{Re}\{\tilde{C}_{ZZ}\} \right] \]

\[ C_{XY} = \frac{1}{2} \left[ - \text{Im}\{C_{ZZ}\} + \text{Im}\{\tilde{C}_{ZZ}\} \right] \]

\[ C_{YX} = \frac{1}{2} \left[ \text{Im}\{C_{ZZ}\} + \text{Im}\{\tilde{C}_{ZZ}\} \right] \]

Thus, \( C_{WW} \) is determined by \( C_{ZZ} \) and \( \tilde{C}_{ZZ} \).

* Therefore, the conclusion follows.

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Notation. (Unconventional) If a complex-valued random vector \( Z \) with mean \( \mu \), covariance \( C \), and pseudo-covariance \( \tilde{C} \) is Gaussian, then it is denoted as

\[ Z \sim \mathcal{CN}(\mu; C, \tilde{C}) \].
Definition. If $Z \sim \mathcal{CN}(\mu; C, \tilde{C})$ has a vanishing pseudo-covariance matrix, i.e., $\tilde{C} = O$, then it is called **proper-complex Gaussian**.

Notation. (Conventional) $Z \sim \mathcal{CN}(\mu; C, O)$ is simply denoted as $Z \sim \mathcal{CN}(\mu, C)$.

Theorem. If $Z \sim \mathcal{CN}(\mu, C)$ has length $N$ and $C \succ 0$, then

$$f_Z(\tilde{z}) \triangleq \frac{1}{\pi^N \det(C)} \exp \left( - (Z - \mu)^H C^{-1} (Z - \mu) \right)$$

$$= f_{X,Y}(x, y)$$

where $Z \triangleq X + jY$.

Proof. Omitted.
* Specialize! $N = 1$. If $Z \sim \mathcal{CN}(0, 2\sigma^2)$, then the joint density of $X$ and $Y$ such that $Z = X + jY$ is given by

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right)$$

- $X \sim \mathcal{N}(0, \sigma^2)$
- $Y \sim \mathcal{N}(0, \sigma^2)$
- $X$ and $Y$ are independent.
- 3 dB problem
- If non-zero mean, then circularly symmetric joint PDF about the mean.
Definition. \( Z \sim \mathcal{CN}(0, C) \) is called a **proper-complex white Gaussian random vector** if there exists \( \sigma^2 \) such that

\[
C = 2\sigma^2 I.
\]

* Note that

\[
\begin{align*}
\text{Cov}[\text{Re}\{Z_i\}, \text{Re}\{Z_j\}] &= \sigma^2 \delta_{i,j} \\
\text{Cov}[\text{Re}\{Z_i\}, \text{Im}\{Z_j\}] &= 0 \\
\text{Cov}[\text{Im}\{Z_i\}, \text{Im}\{Z_j\}] &= \sigma^2 \delta_{i,j}.
\end{align*}
\]

* In other words, \( X \triangleq \text{Re}\{Z\} \) and \( Y \triangleq \text{Im}\{Z\} \) are i.i.d. with

\( \mathcal{N}(0, \sigma^2 I) \).

* Every real and imaginary components are i.i.d. \( \mathcal{N}(0, \sigma^2) \).
- Properties of proper-complex Gaussian random vectors
  - $\mu_Z$ and $C_{ZZ}$ completely characterize the distribution.
  - Theorem. An affine transform of $Z$, e.g.,
    \[
    W = AZ + b \quad A \in \mathbb{C}^{M \times N} \\
    b \in \mathbb{C}^{M \times 1}
    \]
    leads to a proper-complex Gaussian random vector such that
    \[
    W \sim \mathcal{CN}(A\mu_Z + b, AC_{ZZ}A^H)
    \]
  - Proof.
    - Gaussianity can be justified by considering equivalent real-valued Gaussian random vector.
    - mean vector
    - covariance matrix
    - complementary covariance matrix
Specialize! Each element of a proper-complex Gaussian random vector is a proper-complex Gaussian random variable.

Specialize! A weighted sum of $Z_i$'s as $W = a^H Z$ is a proper-complex Gaussian random variable having a circularly symmetric (about the mean $a^H \mu_Z$) PDF having independent real and imaginary parts satisfying

$$\text{Re}\{W\} \sim \mathcal{N}\left(\text{Re}\{a^H \mu_Z\}, \frac{1}{2}a^H C_Z Z a\right)$$

$$\text{Im}\{W\} \sim \mathcal{N}\left(\text{Im}\{a^H \mu_Z\}, \frac{1}{2}a^H C_Z Z a\right).$$

Any collection of the elements of a proper-complex Gaussian random vector is a proper-complex Gaussian random vector.
Example 1. When $Z \sim \mathcal{CN}(\mu, 2\sigma^2 I)$, find $\Pr(\|Z - \mu\| \geq \|Z - c\|)$.

Sol. Note that

\[
\Pr(\|Z - \mu\| \geq \|Z - c\|) = \Pr(\|Z - \mu\|^2 \geq \|Z - c\|^2)
\]
\[
= \Pr(\|Z\|^2 - 2 \text{Re}\{\mu^H Z\} + \|\mu\|^2 \geq \|Z\|^2 - 2 \text{Re}\{c^H Z\} + \|c\|^2)
\]
\[
= \Pr\left(\text{Re}\{(c - \mu)^H Z\} \geq \frac{\|c\|^2 - \|\mu\|^2}{2}\right)
\]
\[
= \Pr\left(\text{Re}\{W\} \geq \frac{\|c\|^2 - \|\mu\|^2}{2}\right).
\]

Since the complex Gaussian random variable $W \triangleq (c - \mu)^H Z$ has

\[
\mathbb{E}[W] = (c - \mu)^H \mu
\]
\[
\text{Cov}[W] = (c - \mu)^H (2\sigma^2 I) (c - \mu),
\]
we have

\[
\Pr(\|Z - \mu\| \geq \|Z - c\|)
= Q\left(\frac{\|c\|^2 - \|\mu\|^2}{2\sigma}\right) - \left[\frac{\Re\{c^H \mu\} - \|\mu\|^2}{\sigma\|c - \mu\|}\right)
\]

\[
= Q\left(\frac{\|c - \mu\|^2}{2\sigma\|c - \mu\|}\right) = Q\left(\frac{\|c - \mu\|}{2\sigma}\right).
\]

Fig. (Geometrical interpretation)
Example 2. When $Z \sim \mathcal{CN}(\mu, C) \in \mathbb{C}^N$, find $\Pr(\|Z - \mu\| \geq \|Z - p\|)$.

Sol. Note that

$$\Pr(\|Z - \mu\| \geq \|Z - p\|) = \Pr(\|Z - \mu\|^2 \geq \|Z - p\|^2)$$

$$= \Pr\left(\text{Re}\left((p - \mu)^H Z\right) \geq \frac{\|p\|^2 - \|\mu\|^2}{2}\right).$$

Let $W \triangleq (p - \mu)^H Z$. Then,

$$\mathbb{E}[W] = (p - \mu)^H \mu$$

$$\text{Cov}[W] = (p - \mu)^H C (p - \mu).$$

Thus, we have

$$\Pr(\|Z - \mu\| \geq \|Z - p\|) = \Pr\left(\text{Re}\{W\} \geq \frac{\|p\|^2 - \|\mu\|^2}{2}\right)$$

$$= \Phi\left(\frac{\|p\|^2 - \|\mu\|^2}{\sqrt{\frac{1}{2} (p - \mu)^H C (p - \mu)}}\right) = \Phi\left(\frac{\|p - \mu\|^2}{\sqrt{2 (p - \mu)^H C (p - \mu)}}\right).$$