Complex Baseband Representation of Real Band-pass Signals: Main Result

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- Title dissection
  - Representation
    * $x(t)$: a time function
    * $X(f) \triangleq \mathcal{F}\{x(t)\}$: a frequency domain representation of $x(t)$
  - Complex baseband signal $\equiv$ real bandpass signal
  - In what sense?

- Why real bandpass signals and systems do we deal with?

- What is the advantage of handling complex baseband representation of real bandpass signals and systems?
• Why real bandpass signals?
  – Electrical signals are real-valued.
    \[
    \begin{cases}
      \text{current} \\
      \text{voltage}
    \end{cases}
    \]
  – Fig.
- In many systems, spectrum is a valuable resource. For example, in a cable for CATV, we want to put as many channel signals as possible.
- Fig.
• What is the advantage of handling complex baseband signals instead of real bandpass signals?
  - Two real numbers are equivalent to one complex number
    * There are many cases where handling a complex number is much easier than handling two real numbers.

  - In real bandpass signaling, it can be shown that two real baseband signals contain “all” the information of the real bandpass signal
    \[ x(t) \equiv (x_c(t), x_s(t)) \equiv z(t) \triangleq x_c(t) + jx_s(t) \]
    one real bandpass       two real baseband       one complex baseband
The main result

- Given
  - $x(t)$, a real-valued "bandpass" signal with finite energy and
  - $X(f) \triangleq \mathcal{F}\{x(t)\}$ that has a finite support $f \in (0, B]$,
- we choose $f_c \in (B/2, B)$ and $\theta \in [0, 2\pi)$. ($\leftarrow$ NOT unique!)
- Then, there exists a unique pair $x_c(t)$ and $x_s(t)$ of real-valued band-limited baseband signals such that

$$x(t) = x_c(t) \cos(2\pi f_c t + \theta) - x_s(t) \sin(2\pi f_c t + \theta)$$

$$= \Re\{(x_c(t) + jx_s(t))e^{j(2\pi f_c t + \theta)}\}$$

where $j \triangleq \sqrt{-1}$.

$- e^{j\phi} = \cos \phi + j \sin \phi$  \hspace{1cm} (Euler's identity)
\[-x_c(t) \cos(2\pi f_c t + \theta): \text{in-phase component of } x(t)\]
\[-x_s(t) \sin(2\pi f_c t + \theta): \text{quadrature component of } x(t). \ (2\pi/4 = 90^\circ)\]

- The complex baseband signal \(x_c(t) + jx_s(t)\), the center frequency \(f_c\), and the phase \(\theta\) contain all the information carried by \(x(t)\).

- The complex baseband signal \(x_c(t) + jx_s(t)\) is called the complex envelope of the real bandpass signal \(x(t)\) and often denoted by \(x_l(t)\).

- \(x_c(t)\): real part of \(x_l(t)\). Often called the in-phase component of \(x_l(t)\)
- \(x_s(t)\): imaginary part of \(x_l(t)\). Often called the quadrature component of \(x_l(t)\).
- It is obvious that, if \( x_c(t) \) and \( x_s(t) \) are real-valued baseband signals having bandwidth \( B' \), then \( x(t) \triangleq x_c(t) \cos(2\pi f_c t + \theta) - x_s(t) \sin(2\pi f_c t + \theta) \) is a real-valued bandpass signal for \( f_c > B' \).

- The main result says that a kind of the converse of this statement is also true.

- How can we prove?
  * In the time domain, we have no idea.
  
  We tackle the problem in the frequency domain.